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Numerical calculation of scaling exponents of percolation process in the framework of renormalization group approach

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Abstract. We use the renormalization group theory to study the directed bond percolation (Gribov process) near its second-order phase transition between absorbing and active state. We present a numerical calculation of the renormalization group functions in the ϵ -expansion where ϵ is a deviation from the upper critical dimension $d_c = 4$. Within this procedure anomalous dimensions γ are expressed in terms of irreducible renormalized Feynman diagrams and thus the calculation of renormalization constants could be entirely skipped. The renormalization group is included by means of the R operation, and for computational purposes we choose the null momentum subtraction scheme.

1 Introduction

The renormalization group (RG) method is a theoretical framework which is especially suitable for studying various critical phenomena [1, 2]. From a computational point of view it provides techniques for a perturbative calculation of different critical exponents. One of the most prominent dynamical models [3] which exhibits a second order phase transition is the directed bond percolation [4, 5]. In the physical literature it is known also as Schlögl first reaction [6]. Among others it explains hadron interactions at very high energies (Reggeon field theory) [7], stochastic reaction-diffusion processes on a lattice [8], spreading of infection diseases [9], etc. The critical exponents are calculable in the form of perturbative expansion in a formally small parameter ϵ . We note that two-loop results for the exponents z and δ were obtained in [10] and exponents ν and β later on in [11]. All necessary

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information concerning percolation process in terms of reaction-diffusion model can be found in the review article [9].

As is in detail discussed in the literature [1] (Part 3.5) or [2] (Part 7.5) the central idea behind renormalization group is freedom in choose of particular renormalization scheme. All of them makes a theory finite with respect to ultraviolet divergences and regarding universal quantities they lead to the same result. For practical numerical calculations it is more convenient to choose the subtractions at normalization point $\{p = 0, \omega = 0, \tau = \mu^2\}$ as explained in [1] (see Eq. (3.18)).

Furthermore, in the renormalized Green function it is possible to replace an additional contribution of the renormalized constant by the operator R [1] applied to Green functions

$$\Gamma^R = R\Gamma = (1 - K)R'\Gamma, \quad (1)$$

where R' is the incomplete operator R that cancels divergences in subgraphs of a given graph and the operator $(1 - K)$ eliminates the remaining superficial divergence.

The main of this work is to show main steps of algorithmic procedure, which allows us to reproduce known two-loops results to very high precision. Moreover it easy to generalize our procedure to high orders and thus obtaine more reliable results.

2 Renormalization of the model

A field theoretical formulation of the percolation process [9] is based on the following action

$$S = \psi^\dagger(-\partial_t + D_0\partial^2 - D_0\tau_0)\psi + \frac{D_0\lambda_0}{2}[(\psi^\dagger)^2\psi - \psi^\dagger\psi^2], \quad (2)$$

where ψ is a coarse-grain density of percolating agents, ψ^\dagger is an auxiliary (Martin-Siggia-Rose) response field, D_0 is a diffusion constant, λ_0 is a positive coupling constant and τ_0 is a deviation from the threshold value of injected probability (an analog of critical temperature in static models). The model is studied near its critical dimension $\epsilon = 4 - d$ in the region where τ_0 acquires its critical value. The expansion parameter of the perturbation theory is rather λ_0^2 than λ_0 as it could be easily seen by a direct inspection of Feynman diagrams. Hence it is more convenient to introduce a new charge $u = \lambda^2$. The renormalized action functional can be written in the following form

$$S_R = \psi^\dagger(-Z_1\partial_t + Z_2D\partial^2 - Z_3D\tau)\psi + \frac{Z_4D\lambda\mu^\epsilon}{2}[(\psi^\dagger)^2\psi - \psi^\dagger\psi^2]. \quad (3)$$

It can be shown [9] that this kind of a model is multiplicatively renormalizable. Furthermore the action functional S_R can also be obtained from the action S by the standard procedure of multiplicative renormalization of all the fields and parameters

$$\psi_0 = \psi Z_\psi, \quad \psi_0^\dagger = \psi^\dagger Z_{\psi^\dagger}, \quad D_0 = D Z_D, \quad \lambda_0 = \lambda \mu^\epsilon Z_\lambda, \quad \tau_0 = \tau Z_\tau. \quad (4)$$

The relations between renormalized constants Z_i , $i = 1, 2, 3, 4$ are obtained in a straightforward fashion and read

$$\begin{aligned} Z_1 &= Z_\psi Z_{\psi^\dagger}, & Z_2 &= Z_D Z_\psi Z_{\psi^\dagger}, \\ Z_3 &= Z_D Z_\tau Z_\psi Z_{\psi^\dagger}, & Z_4 &= Z_D Z_\lambda Z_{\psi^\dagger}^2 Z_\psi = Z_D Z_\lambda Z_{\psi^\dagger} Z_{\psi^\dagger}^2 Z_\psi. \end{aligned} \quad (5)$$

Moreover, the relation $Z_\psi = Z_{\psi^\dagger}$ is satisfied. In this work, at the normalization point (NP) $p = 0$, $\omega = 0$ and $\tau = \mu^2$ is considered. The counterterms are then specified at the normalization point (NP),

and it is advantageous to express renormalization constants in terms of normalized Green functions

$$\begin{aligned}\bar{\Gamma}_1 &= \partial_{i\omega} \Gamma_{\psi^\dagger \psi} \Big|_{p=0, \omega=0}, & \bar{\Gamma}_3 &= -\frac{\Gamma_{\psi^\dagger \psi} - \Gamma_{\psi^\dagger \psi} \Big|_{\tau=0}}{D\tau} \Big|_{p=0, \omega=0}, \\ \bar{\Gamma}_2 &= -\frac{1}{2D} \partial_p^2 \Gamma_{\psi^\dagger \psi} \Big|_{p=0, \omega=0}, & \bar{\Gamma}_4 &= \frac{\Gamma_{\psi^\dagger \psi^\dagger \psi} - \Gamma_{\psi^\dagger \psi \psi}}{D\lambda\mu^\epsilon} \Big|_{p=0, \omega=0},\end{aligned}\quad (6)$$

that satisfy the following conditions $\bar{\Gamma}_i^R|_{\tau=\mu^2} = 1$, $i = 1, 2, 3, 4$. RG constants defined by these conditions do not depend on m , like in minimal subtraction (MS) scheme. Accordingly RG equations are the same as in MS scheme

$$(\mu\partial_\mu + \beta_u\partial_u - \tau\gamma_\tau\partial_\tau - D\gamma_D\partial_D)\Gamma_i^R = (n_\psi\gamma_\psi + n_{\psi^\dagger}\gamma_{\psi^\dagger})\Gamma_i^R, \quad (7)$$

where μ is a reference mass scale, n_ψ and n_{ψ^\dagger} are the numbers of the corresponding fields entering the Green function under consideration, $\gamma_x = \mu\partial_\mu \log Z_x$ are anomalous dimensions and $\beta_u = u(-2\epsilon - \gamma_u)$ is a beta function describing a flow of the charge u under the RG transformation [1]. Using these equations we find relations for the normalized functions

$$(\mu\partial_\mu + \beta_u\partial_u - \tau\gamma_\tau\partial_\tau - D\gamma_D\partial_D)\bar{\Gamma}_i^R = \gamma_i\bar{\Gamma}_i^R. \quad (8)$$

Here, anomalous dimensions γ_i are obtained from relations (5) between the renormalization constants

$$\gamma_1 = 2\gamma_\psi, \quad \gamma_3 = 2\gamma_\psi + \gamma_D + \gamma_\tau, \quad \gamma_2 = 2\gamma_\psi + \gamma_D, \quad \gamma_4 = 3\gamma_\psi + \gamma_D + \gamma_\lambda. \quad (9)$$

Taking into account the renormalization scheme we can express the anomalous dimension in terms of the renormalized derivatives of the one-particle irreducible Green function $\bar{\Gamma}_i$ at the normalization point [12–14]

$$F_i \equiv -[\tilde{\tau}\partial_{\tilde{\tau}}\bar{\Gamma}_i^R(\tilde{\tau})] \Big|_{\tilde{\tau}=1}, \quad i = 1, 2, 4 \quad (10)$$

where $\tilde{\tau} = \tau/\mu^2$. At the normalization point ($\tilde{\tau} = 1$), γ_i takes the form [12, 13]

$$\gamma_i = \frac{2F_i}{1 + F_2 - F_3}, \quad i = 1, 2, 4. \quad (11)$$

For later considerations it is reasonable to introduce new functions (see [13])

$$f_i \equiv R[-\tilde{\tau}\partial_{\tilde{\tau}}\bar{\Gamma}_i(\tilde{\tau})] \Big|_{\tilde{\tau}=1}. \quad (12)$$

These functions are related to the functions F_i (10) in the following way

$$f_i - F_i = f_i F_3, \quad i = 1, 2, 4. \quad (13)$$

We rewrite equations in (11) to obtain relations for anomalous dimensions in terms of the renormalized derivatives of the one-irreducible Green function $\bar{\Gamma}_i$ with respect to $\tilde{\tau}$ at the normalized point

$$\gamma_i = \frac{2f_i}{1 + f_2}, \quad i = 1, 2, 4. \quad (14)$$

The main benefit of this procedure considering (12) is that the operator R is taken at the normalization point and it can be expressed in terms of a subtracting operator $1 - K_i$ that eliminates all divergences from the Feynman graphs [1]

$$R\Gamma = \prod_i (1 - K_i)\Gamma, \quad (15)$$

where the product is taken over all relevant subgraphs of the given Feynman graph, including also the graph as a whole. In the NM scheme we obtain the following representation for the R -operator [12, 14]

$$R\chi = \prod_i \frac{1}{n_i!} \int_0^1 da_i (1 - a_i)^{n_i} \partial_{a_i}^{n_i+1} \chi(\{a\}), \quad (16)$$

where the product is taken over all one-irreducible subgraphs χ_i (again including the graph χ as a whole) with the canonical dimension $n_i \geq 0$ and a_i is a parameter that stretches momenta flowing into the i -th subgraph inside this graph. The main outcome of this approach is that integrals are finite for $\epsilon = 0$. This scheme enables us to calculate a contribution from each diagram to counterterms [12]

$$Z_i = \frac{2}{n\epsilon} [f_i - J\bar{\Gamma}_i^{(n)}], \quad (17)$$

where n is a number of loops. The second term on the RHS stands for a sum of diagrams of lower order perturbation theory. This allows us to recursively calculate counterterms in the NM scheme at the normalized point and thus gives us an opportunity to compare the results with ones obtained in the MS scheme.

3 Calculation of anomalous dimensions

In this part, we illustrate the method described in the previous section by its application to a specific diagram. Let us consider a two-loop contribution to f_3 from (12) determined by the two-loop three-point diagram of Γ_4 with the dimension $n_\chi = 0$. The diagram has one relevant subgraph: the

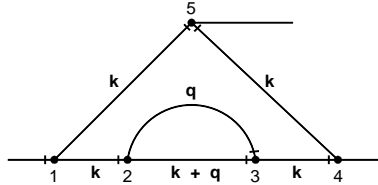


Figure 1. The two-loop Feynman diagram of Γ_4 with symmetry factor equal to one.

subgraph given by vertices $\{2, 3\}$ with the dimension $n_a = 2$.

The action of the differential operator $-\tilde{\tau}\partial_{\tilde{\tau}}$ on the line $G(k) = 1/(k^2 + \tilde{\tau})$ produces an additional factor $1/(k^2 + \tilde{\tau})$ and it corresponds to the insertion of a unit vertex into the propagator line. Graphically it will be denoted by the additional two-point interaction vertex. The application of the operator to the diagram results into a sum over all possible insertions of the vertex

$$-\tilde{\tau}\partial_{\tilde{\tau}}\chi = \begin{array}{c} \text{Diagram 1: Triangle with vertices 1, 2, 3, 4, 5. A unit vertex (black dot) is inserted on the line between 2 and 3.} \end{array} + \begin{array}{c} \text{Diagram 2: Triangle with vertices 1, 2, 3, 4, 5. A unit vertex (black dot) is inserted on the line between 3 and 4.} \end{array} + \dots \quad (18)$$

The next step consists of the inclusion of the operator R , but the analysis of each diagram has to be made separately. For example for the first diagram on the RHS of Eq.(18), with the point outside the subgraph, nothing changes the dimension of the subgraph and $n_a = 2$ remains valid. On the other

hand, in the second graph, the subgraph becomes logarithmic and the corresponding dimension is changed to $n_a = 0$. In this way we obtain the expansion

$$R(-\tilde{\tau}\partial_{\tilde{\tau}}\chi)\big|_{\tilde{\tau}=1} = \frac{1}{2} \int_0^1 da (1-a)^2 \partial_a^3 + \int_0^1 da \partial_a + \dots \quad (19)$$

Further it is necessary to multiply all external parameters for a given subgraph by the parameter a . For the propagator line this means that $G(q+k) = 1/[(aq+k)^2 + 1]$.

Combining (9) and (12) we can derive relations for anomalous dimensions γ for fields and parameters of the model

$$\gamma_\psi = \frac{f_1}{1+f_2}, \quad \gamma_D = \frac{2(f_2-f_1)}{1+f_2}, \quad \gamma_u = 2 \frac{2f_4-f_1-2f_2}{1+f_2}, \quad (20)$$

where f_i s are given up to the two-loop approximation by the following expressions:

$$f_1 = -\frac{u}{16} + \frac{u\epsilon}{32} + 0.0152772u^2, \quad f_2 = -\frac{u}{32} + \frac{u\epsilon}{64} + 0.0062804u^2, \quad f_4 = -\frac{u}{4} + \frac{u\epsilon}{8} + 0.117185u^2. \quad (21)$$

These results were obtained by a numerical calculation in which the actual form of integrals is determined by the R operator using the Feynman representation. Subsequently, the momentum integrals are calculated by the Monte Carlo methods [15]. To the second order of perturbation theory, there are 2 diagrams for the function $\Gamma_{\psi^\dagger\psi}$ and 11 diagrams for the function $\Gamma_{\psi^\dagger\psi^2}$.

The scaling regimes are associated with the fixed points (FPs) of the RG transformation. The asymptotic large scale behavior is governed by the infrared fixed points. Their coordinates can be found from the requirement that β -functions vanish. The directed bond percolation process has only one β -function

$$\beta_u = u(-2\epsilon - \gamma_u) \approx u \left(-2\epsilon + \frac{3u}{4} - \frac{3u\epsilon}{8} - \frac{3u^2\epsilon}{128} - 0.389626u^2 \right). \quad (22)$$

There are two FPs given by the equation above: the trivial (Gaussian or free) FP ($u = 0$) and the non-trivial one of the following form:

$$u^* = \frac{8}{3}\epsilon + 5.02756\epsilon^2 + O(\epsilon^3). \quad (23)$$

that corresponds to the critical percolation process. After the determination of the FP coordinates, critical exponents can be analyzed. First, the critical exponent η takes the following value

$$\eta \equiv 2\gamma_\psi\big|_{u=u^*} = -\frac{\epsilon}{3} - 0.27228\epsilon^2 + O(\epsilon^3). \quad (24)$$

The second is the so-called dynamical exponent z which is associated with the distinctive behavior with respect to the time direction

$$z \equiv 2 - \gamma_D\big|_{u=u^*} = 2 - \frac{\epsilon}{6} - 0.11682\epsilon^2 + O(\epsilon^3). \quad (25)$$

Needed momentum integrals were calculated with the numerical precision of 10^{-4} . For comparison with an analytic calculation we report the appropriately changed results (rescaling $\epsilon \rightarrow 2\epsilon$ is needed)

from [9, 11]

$$\eta = -\frac{\epsilon}{3} \left[1 + \left(\frac{25}{144} + \frac{161}{72} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \approx -\frac{\epsilon}{3} - 0.2723000633\epsilon^2 + O(\epsilon^3), \quad (26)$$

$$z = 2 - \frac{\epsilon}{6} \left[1 + \left(\frac{67}{144} + \frac{59}{72} \ln \frac{4}{3} \right) \epsilon + O(\epsilon^2) \right] \approx 2 - \frac{\epsilon}{6} - 0.1168362090\epsilon^2 + O(\epsilon^3). \quad (27)$$

We thus obtain excellent agreement with our results (24) and (25).

Our two-loop results are in agreement with analytic calculations and our numerical method is suitable for the calculation of the Feynman graphs to the three-loop order. To this end, it is necessary to take into account altogether 17 graphs for the one-irreducible Green function $\Gamma_{\psi^i\psi}$ and 150 graphs for the function $\Gamma_{\psi^i\psi^2}$. It is also feasible to use this method for higher-loop computations. One just has to keep in mind that in order to achieve the required accuracy, computer time needed for calculation of each of the diagrams is much longer. The work in this direction is still in progress.

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